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ELEMENTARY FIRST INTEGRALS OF DIFFERENTIAL EQUATIONS

BY

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ABSTRACT. We show that if a system of differential equations has an elementary first integral (i.e. a first integral expressible in terms of exponentials, logarithms and algebraic functions) then it must have a first integral of a very simple form. This unifies and extends results of Mordukhai-Boltovski, Ritt and others and leads to a partial algorithm for finding such integrals.

1. Introduction. It is not always possible and sometimes not even advantageous to write the solutions of a system of differential equations explicitly in terms of elementary functions. Sometimes, though, it is possible to find elementary functions that are constant on solution curves, that is, elementary first integrals. These first integrals allow one to occasionally deduce properties that an explicit solution would not necessarily reveal. Consider the following example:

EXAMPLE 1. The preditor-prey equations

$$\frac{dx}{dt} = ax - bxy, \quad \frac{dy}{dt} = -cy + dxy, \qquad a, b, c, d \text{ positive real numbers.}$$

Although these cannot be solved explicitly in finite terms, one can show that

$$F(x, y) = dx + by - c\log x - a\log y$$

is constant on solution curves $(x(t), y(t))$. Using the function $F(x, y)$, one can furthermore show that all solution curves in the positive quadrant are closed, that is, all such solutions are periodic.

Note that in this example the first integral is of the form

$$w_0(x, y) + \sum c_i \log w_i(x, y),$$

where the c_i are constants and the w_i are algebraic (in this case, even rational) functions of x and y. Roughly speaking, the main result of this paper is that if a system of differential equations has an elementary first integral, it will then have one of this form. Corollaries of the main result will show that the theory presented here unifies and generalizes a number of results originally due to Mordukhai-Boltovski, Ritt and others. An attempt to do this was made in **[SING: 77]** but the results presented here are more general and the techniques more to the point. Some of these results also appear in **[PRELLE: 82]**. In the following, **Z** stands for the integers, **Q** the rationals and **C** the complex numbers.

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2. Main result and corollaries. To fix notation, we let (K, Δ) denote a differential field of characteristic zero with a given set of derivations $\Delta = \{\delta\}_{\delta \in \Delta}$. The constants of (K, Δ), that is, all those elements annihilated by all δ in Δ, will be denoted by $C(K, \Delta)$. We assume that the reader is familiar with the definitions of elementary and liouvillian extensions and related notions. For precise definitions see [**ROS: 76** or **ROSSIN: 77**].

The following is our main result. Note that, for $\delta_1, \ldots, \delta_n$ in Δ, any K linear combination, $y_1\delta_1 + \cdots + y_n\delta_n$, $y_i \in K$, is a derivation on any Δ-differential extension of K.

THEOREM. *Let (L, Δ) be an elementary extension of the differential field (K, Δ) with $C(L, \Delta) = C(K, \Delta)$. Let $D = y_1\delta_1 + \cdots + y_n\delta_n$ for some $\delta_i \in \Delta$ and $y_i \in K$ and assume that $C(L, \Delta)$ is a proper subset of $C(L, \{D\})$. Then there exist elements of L, $w_0, w_1, \ldots, w_m$, algebraic over K and $c_1, \ldots, c_m$ in $C(K, \Delta)$ such that*

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0 \quad \text{and} \quad \delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0$$

for some $\delta \in \Delta$.

Let us see how Example 1 fits into this scheme.

EXAMPLE 1 REVISITED. Let $K = \mathbf{C}(x, y)$, $\mathbf{C}$ being the complex numbers, x and y indeterminants. Let $\Delta = \{\delta_x, \delta_y\}$ where δ_x (resp. δ_y) is the partial derivative with respect to x (resp. y). Let $D = (ax - bxy)\delta_x + (-cy + dxy)\delta_y$. Let (L, Δ) be an elementary extension of (K, Δ). L then consists of elementary functions of two variables. For g in L, $Dg = 0$ is equivalent to g being constant on solutions of our system of differential equations. For g in L, $\delta_x g \neq 0$ or $\delta_y g \neq 0$ is equivalent to g being not identically constant. Therefore, the hypothesis that $C(L, \Delta)$ is properly contained in $C(L, \{D\})$ is equivalent to the existence of a nonconstant elementary function of two variables that is constant on solutions of our equation. The conclusion states that there must exist $w_0, w_1, \ldots, w_m$ algebraic over $\mathbf{C}(x, y)$ such that $w_0 + \sum_{i=1}^{m} c_i \log w_i$ is constant on solutions of our system (since

$$D\left(w_0 + \sum_{i-1}^{m} c_i \log w_i\right) = Dw_0 + \sum c_i \frac{Dw_i}{w_i} = 0\Bigg)$$

and such that $w_0 + \sum_{i=1}^{m} c_i \log w_i$ is not identically constant. Notice that letting $m = 2$, $w_0 = dx + by$, $c_1 = -c$, $w_1 = x$, $c_2 = -a$, and $w_2 = y$ illustrates the conclusion of the Theorem.

This example may lead one to conjecture that the $w_0, w_1, \ldots, w_m$ guaranteed to exist by the Theorem may be chosen to actually lie in K, rather than being just algebraic over K. This is not necessarily true, even if K is a liouvillian extension of its field of contants.

EXAMPLE 2. Let $k = \mathbf{C}(x)$ with the usual derivation with respect to x which we denote by $'$. Let $E = k\langle \sin^{-1} x \rangle = k((1 - x^2)^{1/2}, \sin^{-1} x)$ and $K = k\langle y \rangle$ where $y = (1 - x^2)^{1/2} \sin^{-1} x$. Notice that y is not algebraic over k and that $y' = 1 - xy/(1 - x^2)$ so $K = k(y)$. K is therefore a purely transcendental extension of $\mathbf{C}$

which can be made into a Δ-differential field by letting $\Delta = \{\delta_x, \delta_y\}$ where δ_x (resp. δ_y) is just the usual partial derivative with respect to x (resp. y). Furthermore, K is a Δ-liouvillian extension of $\mathbf{C}$. Letting $D = \delta_x + (1 - xy/(1 - x^2))\delta_y$, we see that D and $'$ agree on K. E is an algebraic (and therefore elementary) extension of K. In E we have

$$D\frac{y}{\sqrt{1-x^2}} - i\frac{D\left(x + i\sqrt{1-x^2}\right)}{x + i\sqrt{1-x^2}} = 0$$

while

$$\delta_y\frac{y}{\sqrt{1-x^2}} - i\frac{\delta_y\left(x + i\sqrt{1-x^2}\right)}{x + i\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \neq 0.$$

Recall that

$$\int\frac{1}{\sqrt{1-x^2}} = \sin^{-1}x = i\ln\left(x + i\sqrt{1-x^2}\right).$$

Therefore the conclusion of the Theorem is satisfied. Yet it is not true that there exist $w_0, w_1, \ldots, w_m$ in K, such that

$$(1) \qquad Dw_0 + \sum c_i\frac{Dw_i}{w_i} = 0 \quad \text{and} \quad \delta w_0 + \sum c_i\frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta.$$

In fact, it is shown in [**ROSSIN:** 77, p. 335] that if (1) holds for $w_0, w_1, \ldots, w_m$ in K, then $w_0, w_1, \ldots, w_m$ are actually in $\mathbf{C}$ so each $\delta w_i = 0$ for $\delta \in \Delta$.

PROOF OF THE THEOREM. We shall prove a seemingly stronger statement: Let (K, Δ), (L, Δ) and D be as in the Theorem. Assume that there exist $u_0, \ldots, u_m$ in L and $d_1, \ldots, d_m$ in $C(U, \Delta)$ such that

$$(2) \qquad \begin{gathered} Du_0 + \sum_{i=1}^m d_i\frac{Du_i}{u_i} = 0 \quad \text{and} \\ \delta u_0 + \sum_{i=1}^m d_i\frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \in \Delta. \end{gathered}$$

Then there exist $w_0, \ldots, w_n$ in L, algebraic over K, and $c_1, \ldots, c_n$ in $C(K, \Delta)$ such that

$$Dw_0 + \sum_{i=1}^m c_i\frac{Dw_i}{w_i} = 0$$

and

$$\delta w_0 + \sum_{i=1}^m c_i\frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta.$$

The hypotheses of statement (2) are certainly satisfied if the hypotheses of the Theorem are satisfied. Conversely, if the hypotheses of statement (2) are satisfied then in some elementary extension of L, $w = u_0 + \sum_{i=1}^m c_i \log u_i$ satisfies $Dw = 0$ and

$\delta w \neq 0$ for some $\delta \in \Delta$. Yet it is more convenient to prove statement (2). By induction on the transcendence degree of L over K, we may assume that L is an algebraic extension of $K(t)$ where t is transcendental over K and either $\delta t = (\delta v)t$ for some v in K and all δ in Δ or $\delta t = \delta v/v$ for some $v \neq 0$ in K and all δ in Δ.

First assume that there is a w in L such that $Dw = 0$ and w is not in K. We then must have $\delta w \neq 0$ for some δ in Δ since $C(K, \Delta) = C(L, \Delta)$. Since L is algebraic over $K(t)$, w is a root of an irreducible polynomial $W^n + a_{n-1}W^{n-1} + \cdots + a_0$ with the a_i in $K(t)$. If $\delta a_i = 0$ for each δ in Δ and each i, $0 \leqslant i \leqslant n-1$, we would have $\delta w = 0$ for each δ in Δ. Therefore, for some i, $\delta a_i \neq 0$. Similarly, if $Da_i \neq 0$ for some i, we would have $Dw \neq 0$, so we have $Da_i = 0$ for all i. Therefore there exists an element w in $K(t)$ such that $Dw = 0$ and $\delta w \neq 0$ for some $\delta \in \Delta$. If w is in K, we would satisfy the conclusion of (2), so we can assume that w is not in K. If $\delta t = (\delta v)t$ for some v in K and all $\delta \in \Delta$, we have $Dt = (Dv)t$. Since $C(K, \{D\}) \subsetneq C(K(t), \{D\})$, Proposition 1.26 of **[RISCH: 69]** tells us that there exists an integer n and an element s in K such that $Ds = n(Dv)s$. If $\delta s = n(\delta v)s$ for all δ in Δ, we would have $\delta(st^{-n}) = 0$ for all δ in Δ, which would imply that t is algebraic over K, a contradiction. Letting $w_0 = nv$, $w_1 = s$ and $c_1 = -1$, we have $Dw_1 + c_1 Dw_1/w_1 = 0$ and $w_1 + c_1\delta w_1/w_1 \neq 0$ for some δ in Δ which gives the conclusion of (2). If $\delta t = \delta v/v$ for some $v \neq 0$ in K and all δ in Δ, then $Dt = Dv/v$. Again, since $C(K, \{D\}) \subsetneq C(K(t), \{D\})$, Proposition 1.2a of **[RISCH: 69]** tells us that there exists an s in K such that $Ds = Dv/v$. If $\delta s = \delta v/v$ for all δ in Δ, we would have that $\delta(t - s) = 0$ for all δ in Δ. Since t is not algebraic over K, we must have $\delta s \neq \delta v/v$ for some δ in Δ. Letting $w_0 = s$, $w_1 = v$ and $c_1 = -1$, we have $Dw_0 + c_1 Dw_1/w_1 = 0$ and $\delta w_0 + c_1\delta w_1/w_1 \neq 0$ for some δ in Δ, which gives the conclusion of (2).

Now assume that if w is in L and $Dw = 0$ then w is in K, that is, $C(K, \{D\}) = C(L, \{D\})$. Assume also that the hypotheses of (2) are satisfied. We may furthermore assume that the d_i are linearly independent over $\mathbf{Q}$ (otherwise let $e_1, \ldots, e_k$ be a $\mathbf{Q}$-basis of $\mathbf{Q}d_1 + \cdots + \mathbf{Q}d_m$ such that $d_i = (1/\nu)\sum_{j=1}^{k} \nu_{ij}e_j$ with ν_{ij} and ν in $\mathbf{Z}$. We then have

$$0 = Du_0 + \sum_{i=1}^{m} d_i \frac{Du_i}{u_i} = Du_0 + \frac{1}{\nu}\sum_{i=1}^{k} e_i \frac{D(u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}})}{u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}}},$$

$$0 \neq \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} = \delta u_0 + \frac{1}{\nu}\sum_{i=1}^{k} e_i \frac{\delta(u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}})}{u_1^{\nu_{i1}} \cdots u_m^{\nu_{im}}}$$

and we may use these equations instead of those in the hypotheses of (2)). If we have $\delta t = (\delta v)t$ for all δ in Δ, we then have that $Dt = (Dv)t$. Applying Theorem 2 of **[ROS: 76]**, we have that u_0 is algebraic over K and that there exist integers $\nu_0, \nu_1, \ldots, \nu_n$ with $\nu_0 \neq 0$ such that each $u_i^{\nu_i}/t^{\nu_0}$ is algebraic over K. Let

$$w_0 = u_0 + \frac{1}{\nu_0}\sum_{i=1}^{m} d_i \nu_i w_i, \qquad w_i = \frac{u_i^{\nu_0}}{t^{\nu_0}} \quad \text{for } i = 1, \ldots, m, \qquad c_i = \frac{1}{\nu_0} d_i.$$

We then have

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = Du_0 + \sum_{i=1}^{m} d_i \frac{Du_i}{u_i} = 0$$

while

$$\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} = \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

This gives the conclusion of (2). If $\delta t = \delta v/v$ for all δ in Δ, then $Dt = Dv/v$. Applying Theorem 2 of [**ROS: 76**] again we have that $u_1, \ldots, u_m$ are algebraic over K, and that there exists a c in K such that $Dc = 0$ and such that $u_0 - ct$ is algebraic over K. If $\delta c \neq 0$ for some δ in Δ we would be done so we can assume c is in $C(K, \Delta)$. Let

$$w_0 = u_0 - ct, \quad w_i = u_i \quad \text{for } i = 1, \ldots, m,$$
$$w_{m+1} = v, \quad c_i = d_i \quad \text{for } i = 1, \ldots, m,$$
$$c_{m+1} = c.$$

We then have

$$Dw_0 + \sum_{i=1}^{m+1} c_i \frac{Dw_i}{w_i} = Du_0 + \sum_{i=1}^{m} c_i \frac{Du_i}{u_i} = 0$$

while

$$\delta w_0 + \sum_{i=1}^{m+1} c_i \frac{\delta w_i}{w_i} = \delta u_0 + \sum_{i=1}^{m} d_i \frac{\delta u_i}{u_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

This gives the conclusion of (2) and finishes the proof.

We will now deduce some corollaries. Corollary 1 is a generalization of a theorem of Mordukhai-Boltovski [**M-B: 06**] (also see [**RITT: 48**]), which states: Let $y' = f(x, y)$ be a differential equation with f an algebraic function of x and y. If there exists an elementary function $g(x, y)$ which is constant on solutions of $y' = f(x, y)$, then there exist algebraic functions of two variables $\phi_0, \ldots, \phi_m$ and constants $c_1, \ldots, c_m$ such that $\phi_0(x, y) + \sum_{i=1}^{m} c_i \phi_i(x, y)$ is a first integral of $y' = f(x, y)$, that is, it is not identically constant but is constant on all solutions of $y' = f(x, y)$. By a *differential field of functions in $n + 1$ variables $x_0, x_1, x_2, \ldots, x_n$*, we mean a field of functions, meromorphic in some domain in $\mathbf{C}^{n+1}$, closed under the derivations $\partial/\partial x_i$ and containing the coordinate functions $x_1, \ldots, x_n$.

COROLLARY 1. *Let K be a differential field of functions in $n + 1$ variables and L an elementary extension of K. Let f be in K and assume there exists a nonconstant g in L such that g is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$. Then there exist $w_0, \ldots, w_m$ algebraic over K and constants $c_1, \ldots, c_n$ such that*

$$w_0(x, y, y', \ldots, y^{(n-1)}) + \sum_{i=1}^{m} c_i \log w_i(x, y, y', \ldots, y^{(n-1)})$$

is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

PROOF. Let $D = \partial/\partial x_0 + x_2 \partial/\partial x_1 + x_3 \partial/\partial x_2 + \cdots + f \partial/\partial x_n$ and apply the Theorem, noting that $Dg = 0$ if and only if g is constant on all solutions of $y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$.

Loosely speaking, the next result says that if $u_1, \ldots, u_n$ are elementary functions of a single variable x and $g(x, U_1, \ldots, U_n)$ is an elementary function of $n + 1$ variables

$x, U_1, \ldots, U_n$, such that $g(x, \int u_1\,dx, \ldots, \int u_n\,dx)$ is constant, then some nontrivial linear combination with constant coefficients of the $\int u_i$ is elementary. For $n = 1$, this result is due to **[RITT: 23, RITT: 48]**. The result also appears in **[SING: 77]** and includes the result of **[MOZI: 79]**. To be precise, we let k be an ordinary differential field with derivation δ and let $u_1, \ldots, u_n$ be elements of k. Define a new differential field (K, Δ) as follows: let $K = k(U_1, \ldots, U_n)$ where $U_1, \ldots, U_n$ are indeterminants. Let $\Delta = \{\delta_0, \ldots, \delta_n\}$ where:

(1) δ_0 restricted to k is δ and $\delta_0 U_i = 0$ for $i = 1, \ldots, n$.

(2) For $i = 1, \ldots, n$, let $\delta_i a = 0$ for all a in k and let $\delta_i U_i = 0$ if $i \neq j$ and $\delta_i U_i = 1$.

COROLLARY 2. *Let (K, Δ) be as above and let (L, Δ) be an elementary extension of (K, Δ) so that $C(L, \Delta) = C(K, \Delta)$. Let $D = \delta_0 + \sum_{i=1}^{n} u_i \delta_i$ and assume $C(L, \Delta)$ is properly contained in $C(L, \{D\})$. Then there exist $v_0, \ldots, v_m$ in k, and constants $c_1, \ldots, c_n, d_1, \ldots, d_m$ in $C(k, \Delta)$, not all the c_i's being zero, so that*

$$\sum_{i=1}^{n} c_i u_i = \delta v_0 + \sum_{i=1}^{m} d_i \frac{\delta v_i}{v_i}.$$

To prove this corollary we need the following lemma. We assume that the reader is familiar with the notation of Theorem 1 of **[ROS: 76]**.

LEMMA. *Let (K, Δ) and D be as above and let (E, Δ) be an algebraic extension of K with $C(E, \Delta) = C(K, \Delta)$. If there exists an $\alpha \in E$, such that $\alpha \notin k$ and $D\alpha = 0$ then there exist $c_1, \ldots, c_n \in C(k, \{\delta\})$ and $w \in k$ such that $\sum_{i=1}^{n} c_i u_i = \delta w$.*

PROOF. Let $\alpha \in E$ such that $\alpha \notin k$ and $D\alpha = 0$. Let $x^m + a_{m-1}x^{m-1} + \cdots + a_0$ be the minimum polynomial of α over $k(U_1, \ldots, U_n)$. Since $D\alpha = 0$, we have $Da_i = 0$ for $i = 0, \ldots, m-1$. If each $a_i \in k$, then α would be algebraic over k and so $0 = D\alpha = \delta\alpha$ and $\delta_i \alpha = 0$ for $i = 1, \ldots, n$. Since $C(E, \Delta) = C(K, \Delta) \subset k$, α would be in k, a contradiction. We can conclude that for some a_i, $a_i \notin k$ and $Da_i = 0$, and therefore we can assume that $\alpha \in k(U_1, \ldots, U_n)$. Proceeding by induction on n, we can assume that $C(k, \{D\}) = C(k(U_1, \ldots, U_{n-1})\{D\})$ while $C(k, \{D\}) \subsetneq C(k(U_1, \ldots, U_n), \{D\})$. Since $DU_n = u_n$ which is in $k(U_1, \ldots, U_{n-1})$, we can conclude that there is an element V in $k(U_1, \ldots, U_{n-1})$ such that $DV = u_n$. Applying Theorem 1 of **[ROS: 76]** to

$$\begin{gathered} DU_1 \in k \\ \vdots \\ DU_{n-1} \in k \\ DV \in k \end{gathered}$$

while noting that $\operatorname{tr\,deg} k(U_1, \ldots, U_{n-1}, V)/k = n - 1$ we can conclude that the n elements $dU_1, \ldots, dU_{n-1}$, dV of $\Omega_{k(U_1, \ldots, U_{n-1}, V)/k}$ are linearly dependent over $C(k, \{D\}) = C(k, \{\delta\})$. Therefore, for some $c_1, \ldots, c_n$ in $C(k, \{\delta\})$ we have $0 = c_1 dU_1 + \cdots + c_{n-1} dU_{n-1} + c_n dV = d(c_1 U_1 + \cdots + c_n V)$. Therefore, $w = c_1 U_1 + \cdots + c_n V$ is algebraic over k and so in k. We also have

$$D(c_1 U_1 + \cdots + c_n V) = c_1 u_1 + \cdots + c_{n-1} u_{n-1} + c_n u_n = Dw = \delta w.$$

PROOF OF COROLLARY 2. We can conclude from the Theorem that there are $w_0,\dots,w_n$ in L, algebraic over K, such that

$$\begin{aligned} &Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0, \\ (3)\qquad &\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \in \Delta. \end{aligned}$$

As in the proof of the Theorem we can assume that the c_i are linearly independent over $\mathbf{Q}$.

Letting $E = K(w_0,\dots,w_n)$, we see that E is a differential extension of K. Furthermore, we can assume that $C(E,\{D\}) = C(k,\{D\})$, since otherwise the Lemma would imply we were done. We now apply Theorem 1 of **[ROS: 76]** to the $n+1$ equations

$$\begin{aligned} &DU_1 \in k \\ &\vdots \\ &DU_n \in k \\ &\sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} + Dw_0 \in k \end{aligned}$$

and noting that $\operatorname{tr\,deg} k(U_1,\dots,U_n,w_0,\dots,w_n)/k < n+1$, conclude that there exist constants $f_i \in C(k,\{D\})$ such that

$$f_1 dU_1 + \cdots + f_n dU_n + f_{n+1}\left(\sum_{i=1}^{m} c_i \frac{dw_i}{w_i} + dw_0\right) = 0.$$

If $f_{n+1} = 0$, we have $d(f_1U_1 + \cdots + f_nU_n) = 0$, so $v_0 = f_1U_1 + \cdots + f_nU_n$ is algebraic over k (and therefore in k, since k is relatively algebraically closed in K). Therefore

$$f_1u_1 + \cdots + f_nu_n = D(f_1U_1 + \cdots + f_nU_n) = Dv_0 = \delta v_0$$

which gives us the conclusion of the corollary.

If $f_{n+1} \neq 0$, then we can conclude that $f_1U_1 + \cdots + f_nU_n + f_{n+1}w_0$ and all the w_i, $i = 1,\dots,m$, are algebraic over k. For some i, $1 \leqslant i \leqslant n$, we must have $f_i \neq 0$. If not, we would have $w_0, w_1,\dots,w_n$ algebraic over k. Since D restricted to k is δ and each δ_j, $1 \leqslant i \leqslant n$, restricted to k is 0, we have $Dw_i = \delta w_i$ for $i = 0,\dots,m$ and $\delta_j w_i = 0$ for $1 \leqslant j \leqslant n$, $0 \leqslant i \leqslant m$. This would contradict the relations in (3). Let $k_1 = k(f_1U_1 + \cdots + f_nU_n + f_{n+1}w_0, w_1,\dots,w_m)$ and let

$$v_0 = \operatorname{Trace}(f_1U_1 + \cdots + f_nU_n + f_{n+1}w_0) \quad \text{and} \quad v_i = \operatorname{Norm}(w_i) \quad \text{for } i = 1,\dots,m,$$

where the Trace and Norm are taken in the field k_1 with respect to k. We then have, for some integer p,

$$\frac{Dv_i}{v_i} = p\frac{Dw_i}{w_i}, \qquad i = 1,\dots,m,$$

$$Dv_0 = pD(f_1U_1 + \cdots + f_nU_n + f_{n+1}w_0) = pf_1u_1 + \cdots + pf_nu_n + pf_{n+1}Dw_0.$$

Since $Dw_0 = -\sum c_i Dw_i/w_i$ we have

$$pf_1u_1 + \cdots + pf_nu_n = Dv_0 - pf_{n+1}Dw_0$$

$$= Dv_0 + pf_{n+1}\left(\sum_{i=1}^{m} c_i \frac{Dw_i}{w_i}\right) = Dv_0 + f_{n+1}\sum_{i=1}^{m} c_i \frac{Dv_i}{v_i}.$$

This gives us the conclusion of Corollary 2.

Finally, note that the results of [**MACK: 76**] show how one can decide if there exist constants $c_1, \ldots, c_n$, not all zero such that $\sum c_i u_i$ has an elementary antiderivative, where the u_i lie in a purely transcendental elementary extension of $C(x)$.

In the next corollary, which generalizes a result in [**SING: 75**], we will focus on the differential equation $y' = f(y)$ where $f(y)$ is a nonzero function of one variable. Loosely speaking this corollary says that if $y' = f(y)$ has an elementary first integral then

$$g(y) = \int \frac{1}{f(y)}\, dy$$

is an elementary function of y. The converse is also true, since if $g(y)$ is an elementary function of y and $y(x)$ is a solution of $y' = f(y)$ then $(g(y(x)))' = 1$ so $g(y) - x$ is an elementary function of x and y which is constant on solutions of $y' = f(y)$. If $f(y)$ is an elementary function of y, the Risch integration algorithm allows us to decide if $g(y)$ is elementary and so allows us to determine if the differential equation $y' = f(y)$ has an elementary first integral.

Suppose we have a differential equation $y' = f(y)$, where $f \neq 0$. We model this situation in the following way: let $(K, \{\delta_y\})$ be a differential field such that K is a field of functions in the single variable y which contains the element $f(y)$ and δ_y is a derivation of K such that $\delta_y(y) = 1$. We can extend K to the field $K(x)$ where x is transcendental over K. We extend δ_y to a derivation on this field by letting $\delta_y(x) = 0$ and define a new derivation δ_x on $K(x)$ by letting $\delta_x(a) = 0$ for all a in K and $\delta_x(x) = 1$. Let $\Delta = \{\delta_x, \delta_y\}$ and $D = \delta_x + f\delta_y$.

COROLLARY 3. *Let $(K(x), \Delta)$ and D be as above and let (L, Δ) be an elementary extension of $(K(x), \Delta)$ such that $C(K(x), \Delta) = C(L, \Delta)$. Furthermore, assume $C(K(x), \{D\})$ is a proper subset of $C(L, \{D\})$. Then there exist $u_0, u_1, \ldots, u_m$ in K, with $u_1, \ldots, u_m$ nonzero and $a_1, \ldots, a_n$ in $C(K, \{\delta_y\})$ such that*

$$\frac{1}{f} = \delta_y u_0 + \sum_{i=1}^{m} a_i \frac{\delta_y u_i}{u_i}.$$

PROOF. By the Theorem we know that there exist $w_0, w_1, \ldots, w_m$ in L, algebraic over $K(x)$ and $c_1, \ldots, c_m$ in $C(K(x), \Delta)$ such that

$$Dw_0 + \sum_{i=1}^{m} c_i \frac{Dw_i}{w_i} = 0$$

while for some δ in Δ

$$\delta w_0 + \sum_{i=1}^{m} c_i \frac{\delta w_i}{w_i} \neq 0.$$

By an argument given previously, we may assume the c_i are linearly independent over $\mathbf{Q}$. Since $\delta x \in K$ for $\delta \in \Delta$, we may apply Theorem 2 of [**ROS: 76**]. We can conclude that $w_1, \ldots, w_m$ are algebraic over K and that there exists a c in $C(K, \Delta) = C(K, \{\delta_y\})$ such that $w_0 - cx$ is algebraic over K. Let $v_0 = w_0 - cx$ and $v_i = w_i$. We then have

$$cDx + Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} = 0 \tag{4}$$

while for some δ in Δ

$$c\delta x + \delta v_0 + \sum_{i=1}^{m} c_i \frac{\delta v_i}{v_i} \neq 0. \tag{5}$$

Now we shall show that $c \neq 0$. Assume not, then

$$\begin{aligned} 0 = Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} &= \delta_x v_0 \\ &+ \sum_{i=1}^{m} c_i \frac{\delta x v_i}{v_i} + f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right) = f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right) \end{aligned}$$

since $\delta_x v_0 = \delta_x v_1 = \cdots = \delta_x v_m = 0$. We therefore have

$$c\delta x + \delta v_0 + \sum_{i=1} c_i \frac{\delta v_i}{v_i} = \delta v_0 + \sum_{i=1}^{m} \frac{\delta v_i}{v_i} = 0$$

for each δ in Δ, contradicting (5).

Using the fact that $D = \delta_x + f\delta_y$ we can rewrite (4) as

$$0 = cDx + Dv_0 + \sum_{i=1}^{m} c_i \frac{Dv_i}{v_i} = c + f\left(\delta_y v_0 + \sum_{i=1}^{m} c_i \frac{\delta_y v_i}{v_i}\right).$$

Let $M = K(v_0, v_1, \ldots, v_m)$ and let

$$\begin{aligned} u_0 &= -(1/c) \text{ Trace } v_0, \\ u_i &= \text{Norm } v_i \quad \text{for } i = 1, \ldots, m, \\ a_i &= -c_i/c \quad \text{for } i = 1, \ldots, m, \end{aligned}$$

where the Norm and Trace are with respect to L over K. We then have

$$\frac{1}{f} = \delta_y u_0 + \sum_{i=1}^{m} a_i \frac{\delta_y u_i}{u_i}$$

with $u_0, \ldots, u_m$ in K and a_i in $C(K, \{\delta_y\})$.

3. Algorithmic considerations. The preceding work was motivated by our desire to develop a decision procedure for finding elementary first integrals. These results show that we need only look for elementary integrals of a prescribed form. In this section we shall discuss the problem of finding an elementary first integral for a two-dimensional autonomous system of differential equations and reduce this problem to that of bounding the degrees of algebraic solutions of this system.

Let $K = \mathbf{C}(x, y)$, where x and y are transcendental over $\mathbf{C}$, and let $\Delta = \{\delta_x, \delta_y\}$, where these derivations are just the usual partial derivatives with respect to x and y. Consider the system of differential equations,

$$\frac{dx}{dt} = P(x, y), \qquad \frac{dy}{dt} = Q(x, y), \tag{6}$$

where P and Q are polynomials in $\mathbf{C}[x, y]$ and let $D = P\delta_x + Q\delta_y$. We say that (6) has an elementary first integral if there exists an elementary extension (L, Δ) of (K, Δ) such that $C(L, \Delta) = C(K, \Delta)$ and $C(L, \Delta) \subsetneq C(L, \{D\})$. The existence of an elementary first integral is intimately related to the existence of an algebraic integrating factor for $Qdx - Pdy$. Without explicitly mentioning this 1-form, the following propositions describe this relationship.

PROPOSITION 1. *If the equations of* (6) *have an elementary first integral, then there exists an element $R \neq 0$ algebraic over K such that $DR = -(\delta_x P + \delta_y Q)R$.*

PROOF. Applying the Theorem of §2, there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $C(K, \Delta)$ such that $Dw_0 + \sum_{i=1}^n c_i Dw_i/w_i = 0$ and $\delta w_0 + \sum_{i=1}^n c_i \delta w_i/w_i \neq 0$ for some δ in Δ. Let

$$R_1 = \delta_x w_0 + \sum_{i=1}^n c_i \frac{\delta_x w_i}{w_i}, \qquad R_2 = \delta_y w_0 + \sum_{i=1}^n c_i \frac{\delta_y w_i}{w_i}.$$

We then have $0 = PR_1 + QR_2$. We may assume that at least one of P or Q is nonzero, say Q. Letting $R = R_1/Q$, we have $R_1 = QR$ and $R_2 = -PR$. Since δ_x and δ_y commute in any algebraic extension of K, we see that $\delta_y(QR) = \delta_x(-PR)$. Carrying out this differentiation gives us that $DR = -(\delta_x P + \delta_y Q)R$.

PROPOSITION 2. *Assume that there exists an element $S \neq 0$, algebraic over K such that $DS = -(\delta_x P + \delta_y Q)S$. Then either*

(i) *there exists a w in K such that $Dw = 0$ and $\delta w \neq 0$ for δ in Δ, or*

(ii) *for any $R \neq 0$ algebraic over K such that $DR = -(\delta_x P + \delta_y Q)R$, there exists a c in $\mathbf{C}$ such that $R = cS$ and furthermore R^n is in K for some n in $\mathbf{Z}$.*

If (i) *holds, then the equations in* (6) *obviously have an elementary first integral. If* (i) *does not hold, then the equations in* (6) *have an elementary first integral if and only if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that*

$$\delta_x w_0 + \sum c_i \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y w_0 + \sum c_i \frac{\delta_y w_i}{w_i} = -SP.$$

PROOF. Let $R \neq 0$ be algebraic over K and satisfy $DR = -(\delta_x P + \delta_y Q)R$. Furthermore, assume (i) does not hold, that is, that $Dw = 0$ implies $w \in \mathbf{C}$ for w in K. Since R/S satisfies $D(R/S)/(R/S) = 0$ we have R/S is in $\mathbf{C}$, so $R = cS$ for some c in $\mathbf{C}$. Let E be a normal algebraic extension of K containing R. For any K-automorphism σ of E, we have $D(\sigma R)/\sigma R = -(\delta_x P + \delta_y Q)$. Summing this relation over all σ in the galois group of E over K, we get

$$\frac{D(\text{Norm } R)}{\text{Norm } R} = -n(\delta_x P + \delta_y Q)$$

for some n in $\mathbf{Z}$. Therefore $D(\text{Norm } R/R^n)/(\text{Norm } R/R^n) = 0$ so R^n is a constant multiple of Norm R and therefore in K.

Now, assume that (6) has an elementary first integral. We have shown in the proof of Proposition 1 that there exist $w_0, \ldots, w_n$ and R algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = RQ, \qquad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -RP$$

and

$$DR = -(\delta_x P + \delta_y Q)R.$$

If (i) does not hold, then by (ii) we have $R = cS$ for some c in $\mathbf{C}$ so

$$\delta_x\left(\frac{w_0}{c}\right) + \sum_{i=1}^{n} \frac{c_i}{c} \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y\left(\frac{w_0}{c}\right) + \sum_{i=1}^{n} \frac{c_i}{c} \frac{\delta_y w_i}{w_i} = -SP.$$

Conversely, if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ \quad \text{and} \quad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP$$

then

$$Dw_0 + \sum_{i=1}^{n} c_i \frac{Dw_i}{w_i} = P(SQ) - Q(SP) = 0$$

while

$$\delta w_0 + \sum_{i=1}^{n} c_i \frac{\delta w_i}{w_i} \neq 0 \quad \text{for some } \delta \text{ in } \Delta.$$

We can then find an elementary extension (L, Δ) of (K, Δ) with $C(L, \Delta) = C(K, \Delta)$ such that L contains an element w with $Dw = 0$ and $\delta w \neq 0$ for some δ in Δ.

Therefore, to decide if (6) has an elementary first integral, we must:

(A) Decide if $Dw = 0$ has a nonconstant solution in $\mathbf{C}(x, y)$ and find one if it does.

(B) If $Dw = 0$ has only constant solutions in $\mathbf{C}(x, y)$, decide if

$$DS = -(\delta_x P + \delta_y Q)S$$

has a nonconstant algebraic solution S with S^n in $\mathbf{C}(x, y)$ for some n in $\mathbf{Z}$ and find one if it does.

(C) If (B) holds, decide if there exist $w_0, \ldots, w_n$ algebraic over K and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ \quad \text{and} \quad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP$$

and find them if they do.

We can solve (C) completely, so we shall deal with it first. Let k be the algebraic closure of $\mathbf{C}(y)$ and let $F = k(x, S)$. Considering F as an ordinary differential field with derivation δ_x, we have $C(F, \{\delta_x\}) = k$. The first step in solving problem (C) is to decide if $SQ = \delta_x u_0 + \sum c_i \delta_x u_i/u_i$ with u_i in F and c_i in k. Since F is algebraic over $k(x)$, a solution of this latter problem was described in **[RISCH: 70]**. If no such u_i and c_i exist, we are done. If u_i and c_i do exist we can assume that the c_i are linearly independent over **Q** and that $\delta_x u_i \neq 0$ for $i = 1, \ldots, n$. Next decide if the c_i are in **C**, i.e. $\delta_y c_i = 0$ for all i. If some $\delta_y c_i \neq 0$, then we claim that for any choice of $w_0, \ldots, w_n$ in F and $d_1, \ldots, d_n$ in k such that $SQ = \delta_x w_0 + \sum d_i \delta_x w_i / w_i$ we have $\delta_y d_i \neq 0$ for some i. To see this assume we have $w_0, \ldots, w_n$ in k and d_i in **C** such that $SQ = \delta_x w_0 + \sum d_i \delta_x w_i/w_i$ and assume that $c_1 \notin \mathbf{C}$. We then have

$$\delta_x(u_0 - w_0) + \sum c_i \frac{\delta_x u_i}{u_i} - \sum d_i \frac{\delta_x w_i}{w_i} = 0.$$

If we extend $c_1, \ldots, c_n$ to a **Q**-basis of $c_1\mathbf{Q} + \cdots + c_n\mathbf{Q} + d_1\mathbf{Q} + \cdots + d_m\mathbf{Q}$ and rewrite the above equation we get

$$\delta_x(u_0 - w_0) + c_1 \frac{\delta_x u_1}{u_1} + \sum c_i \frac{\delta_x v_i}{v_i} = 0,$$

where the v_i are power products of $u_2, \ldots, u_n$, $w_1, \ldots, w_m$. Since all the terms appearing here are algebraic over $k(x)$ and $\delta_x x = 1$, we have that u_i is in k so $\delta_x u_1 = 0$, a contradiction. Therefore if some c_i is not in **C**, we can conclude that (6) does not have an elementary integral and we are done. Therefore assume that we have found $u_0, \ldots, u_n$ algebraic over F and $c_1, \ldots, c_n$ in **C** such that $SQ = \delta_x u_0 + \sum c_i \delta_x u_i / u_i$. Consider the expression

$$I = \delta_y u_0 + \sum c_i \frac{\delta_y u_i}{u_i} + SP.$$

Since δ_x and δ_y commute, we have

$$\delta_x I = \delta_y \left(\delta_x u_0 + \sum c_i \frac{\delta x u_i}{u_i} - SQ \right) = 0.$$

Therefore, I is in k and so in some finite extension of $\mathbf{C}(y)$. Now use **[RISCH: 70]** to decide if there exist $v_0, \ldots, v_m$ algebraic over $\mathbf{C}(y)$ and $d_1, \ldots, d_m$ in **C** such that

$$I = \delta_y v_0 + \sum_{i=1}^{m} d_i \frac{\delta_y v_i}{v_i}.$$

If such elements exist, we then have

$$\delta_x u_0 - \delta_x v_0 + \sum c_i \frac{\delta_x u_i}{u_i} - \sum d_i \frac{\delta_x v_i}{v_i} = \delta_x u_0 + \sum c_i \frac{\delta_x u_i}{u_i} = SQ$$

and

$$\delta_y u_0 - \delta_y v_0 + \sum c_i \frac{\delta_y u_i}{u_i} - \sum d_i \frac{\delta_y v_i}{v_i} = I - SP - \left(\delta_y v_0 + \sum d_i \frac{\delta_y v_i}{v_i} \right) = -SP$$

and so we are done. If no such elements exist, then we claim that there are no elements $w_0, \ldots, w_k$ algebraic over $\mathbf{C}(x, y)$ and $c_1, \ldots, c_k$ in **C** such that

$$\delta_y w_0 + \sum e_i \frac{\delta_y w_i}{w_i} = -SP.$$

If such elements existed, then we would have

$$\delta_y u_0 + \sum c_i \frac{\delta_y u_i}{u_i} + \delta_y w_0 + \sum e_i \frac{\delta_y w_i}{w_i} = I.$$

This implies that I has an antiderivative (with respect to δ_y) in an elementary extension of $\mathbf{C}(x, y)$. Note that I is algebraic over $\mathbf{C}(y)$ and $\mathbf{C}(x, y)$ is a δ_y elementary extension of $\mathbf{C}(y)$ with new constants. The strong Liouville Theorem of [**RISCH: 69**] implies that there exist $v_0, \ldots, v_n$ algebraic over $\mathbf{C}(y)$ and $c_1, \ldots, c_n$ in $\mathbf{C}$ such that $I = \delta_y v_0 + \sum c_i \delta_y v_i / v_i$. This completes the procedure for (C). Note that if S is actually in $\mathbf{C}(x, y)$ then problem (C) always has a positive solution.

We now turn to problems (A) and (B). Let w be an element of $\mathbf{C}(x, y)$ and write $w = \prod_{i=1}^m f_i^{n_i}$ with f_i irreducible in $\mathbf{C}[x, y]$ and n_i in $\mathbf{Z}$. If $Dw = 0$, we then see f_i must divide Df_i for each i. Let S be an element algebraic over $\mathbf{C}(x, y)$ with S^n in $\mathbf{C}(x, y)$ such that $DS = -(\delta_x P + \delta_y Q)S$. Write $S = \prod_{i=1}^m f_i^{r_i}$ with f_i irreducible in $\mathbf{C}[x, y]$ and r_i in $\mathbf{Q}$. Since

$$\sum_{i=1}^{m} r_i \frac{Df_i}{f_i} = \frac{DS}{S} = -\left(\delta_x P + \delta_y Q\right)$$

we again have that each f_i divides Df_i. Results of Darboux [**JOU: 79**, Theorem 3.3, p. 102 and Lemma 3.53, p. 112] imply that the degree of each f_i is bounded. No effective bound is given. This suggests the following problem:

(D) Given P and Q in $\mathbf{C}[x, y]$, let $D = P\delta_x + Q\delta_y$. Given an effective procedure to find an integer N so that if f is irreducible in $\mathbf{C}[x, y]$ and f divides Df, then the degree of f is less than N.

Assuming one has a solution for problem (D), one can solve problems (A) and (B). The results of Darboux quoted above imply that if $f_1, \ldots, f_m$ are irreducible in $\mathbf{C}[x, y]$ and f_i divides Df_i then either $m < ((d+1)d/2) + 2$ where $d = \max(\deg P, \deg Q)$ or there exist integers n_i, not all zero, such that

$$n_1 \frac{Df_1}{f_1} + \cdots + n_m \frac{Df_m}{f_m} = 0.$$

In the latter case, we would have $Dw = 0$ for $w = \prod_{i=1}^m f_i^{n_i}$. To solve problems (A) and (B), note that once we can find N as in (D), the set of coefficients of polynomials f of degree $\leqslant N$ so that f divides Df forms a projective variety which we can construct. We can therefore decide if this variety has fewer than $d(d+1)/2 + 2$ points. If it has more points, then we can find them and construct a w in $\mathbf{C}(x, y)$ such that $Dw = 0$. If it has fewer, then we find all of them, thereby finding all polynomials f_i of degree $\leqslant N$ such that f_i divides Df_i. We then decide if we can find r_i in $\mathbf{Q}$, not all zero such that

$$\sum r_i \frac{Df_i}{f_i} = -\left(\delta_x P + \delta_y Q\right).$$

Therefore we have reduced problems (A) and (B) to (D). When $\max(\deg P, \deg Q) = 1$, a solution to this problem appears in [**JOU: 79**, pp. 8–19]. For $\max(\deg P, \deg Q) > 1$ no solution seems to be known. Partial results appear in

[**PAIN: 72**, vol. I, pp. 173–218, vol. II, pp. 433–458 and **POINC: 34**, vol. III, pp. 32–97]. Yet even without a solution to problem (D), the above suggests a heuristic method. Arbitrarily fix an integer N. Find all irreducible polynomials f of degree $\leqslant N$ such that f_i divides Df_i. If the resulting Df_i/f_i are linearly dependent over $\mathbf{Z}$, we then can find a w such that $Dw = 0$. If the Df_i/f_i are not linearly independent over $\mathbf{Z}$, decide if there exist r_i in $\mathbf{Q}$ such that

$$\sum r_i \frac{Df_i}{f_i} = -\left(\delta_x P + \delta_y Q\right).$$

If such r_i exist, let $S = \prod f_i^{r_i}$ and decide if there exist $w_0, \ldots, w_n$ algebraic over $\mathbf{C}(x, y)$ and constants $c_1, \ldots, c_n$ in $\mathbf{C}$ such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ, \qquad \delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP.$$

EXAMPLE 1 REVISITED AGAIN. We again consider the system $\dot{x} = ax - bxy$ and $\dot{y} = -cy + dxy$. Letting $D = (ax - bxy)\delta_x + (-xy + dxy)\delta_y$ we let $N = 1$ and look for all polynomials $f = \alpha x + \beta y + \gamma$ of degree $\leqslant 1$ such that f divides Df. Since $Df = (\beta d - \alpha b)xy + a\alpha x - c\beta y$, if f divides Df we must have that either $\alpha = 0$ or $\beta = 0$. In both cases we get $\gamma = 0$. Therefore, there are just two first degree polynomials f, x and y such that f divides Df. Furthermore,

$$\frac{Dx}{x} = -by + a \quad \text{and} \quad \frac{Dy}{y} = dx - c.$$

One can check that, unless $b = d = 0$, these are linearly independent over $\mathbf{Z}$. We now solve

$$r_1 \frac{Dx}{x} + r_2 \frac{Dy}{y} = r_1(-by + a) + r_2(dx - c) = -\left(a - by + (-c + dx)\right)$$

and find $r_1 = r_2 = -1$. Let $S = x^{-1}y^{-1}$ and find $w_0, \ldots, w_n$ and $c_1, \ldots, c_n$ constants such that

$$\delta_x w_0 + \sum_{i=1}^{n} c_i \frac{\delta_x w_i}{w_i} = SQ = -\frac{c}{x} + d,$$

$$\delta_y w_0 + \sum_{i=1}^{n} c_i \frac{\delta_y w_i}{w_i} = -SP = -\frac{a}{y} + b.$$

We get $w_0 = dx + by$, $w_1 = x$, $w_2 = y$, $c_1 = -c$, $c_2 = -a$ and so $dx + by - c \log x - a \log y$ is an elementary first integral.

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